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LETTER TO THE EDITOR

Pair correlations in the antiperiodic Ising strip and conformal invariance

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Abstract. An explicit formula for the spatial dependence of the pair correlation function in an Ising strip with antiperiodic boundary conditions at the bulk critical temperature is given. It follows from the conformal-invariance approach of Belavin *et al* and Cardy.

The conformal-invariance approach (Cardy 1986) has yielded a wealth of new information on finite-size effects in critical two-dimensional systems.

In a conformally invariant two-dimensional system, the n-spin correlation function transforms (Cardy 1984a, b, c, 1986) according to

$$\langle\sigma(w_1)\ldots\sigma(w_n)\rangle_{g'}=|\mathrm{d}w(z_1)/\mathrm{d}z_1\ldots\mathrm{d}w(z_n)/\mathrm{d}z_n|^{-\eta/2}\langle\sigma(z_1)\ldots\sigma(z_n)\rangle_g \tag{1}$$

under a conformal mapping generated by the analytic function $w(z)$. Here complex coordinates $z=x+\mathrm{i}y$ and $w=u+\mathrm{i}v$ are used to specify points in the xy and uv planes. The subscripts g and g′ refer to the boundary geometry which, in general, is modified by the conformal transformation.

Inserting into equation (1) the mapping

$$w(z)=\frac{L}{2\pi}\ln z \tag{2}$$

and the bulk two-point function at criticality

$$\langle\sigma(z_1)\sigma(z_2)\rangle_\infty=A|z_1-z_2|^{-\eta} \tag{3}$$

one obtains (Cardy 1984a) the universal functional form

$$\langle\sigma(w_1)\sigma(w_2)\rangle_{\mathrm{P}}=A\left[\left(\frac{L}{2\pi}\right)^2[2\cosh(2\pi(u_1-u_2)/L)-2\cos(2\pi(v_1-v_2)/L)]\right]^{-\eta/2} \tag{4}$$

for the two-point function in the strip geometry $-\infty<u<\infty$, $0<v<L$ with the periodic boundary condition $\sigma(w+\mathrm{i}L)=\sigma(w)$, i.e. cylindrical topology.

Unlike the bulk pair correlation function (3), the pair correlation function of a semi-infinite critical system is not uniquely determined by simple scaling. Cardy (1984b, 1986) has extended the conformal-invariance approach of Belavin *et al* (1984) for calculating bulk correlations to the half-space $y>0$. For the pair correlation function of the semi-infinite Ising model, he finds

$$\langle\sigma(z_1)\sigma(z_2)\rangle_{\mathrm{S}}=A(4y_1y_2)^{-1/8}(\tau^{1/4}\mp\tau^{-1/4})^{1/2} \tag{5a}$$

$$\tau=[(x_1-x_2)^2+(y_1+y_2)^2][(x_1-x_2)^2+(y_1-y_2)^2]^{-1}. \tag{5b}$$

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The upper and lower signs preceding $\tau^{-1/4}$ in equation (5*a*) hold for free and fixed boundary spins, respectively, i.e. for the ordinary and extraordinary transition (Binder 1983, Burkhardt 1985).

To obtain the corresponding formula for an Ising strip with free and fixed boundary spins, one substitutes this result and the mapping

$$w(z)=\frac{L}{\pi}\ln z \tag{6}$$

of the upper half z plane onto the strip $-\infty<u<\infty$, $0<v<L$ into equation (1). The result is

$$\langle\sigma(w_1)\sigma(w_2)\rangle_{\mathrm{F}}=A[(2L/\pi)^2\sin(\pi v_1/L)\sin(\pi v_2/L)]^{-1/8}(\tau^{1/4}\mp\tau^{-1/4})^{1/2} \tag{7a}$$

$$\tau=\frac{\cosh[\pi(u_1-u_2)/L]-\cos[\pi(v_1+v_2)/L]}{\cosh[\pi(u_1-u_2)/L]-\cos[\pi(v_1-v_2)/L]} \tag{7b}$$

for the strip correlation function. Again the upper and lower signs preceding $\tau^{-1/4}$ in equation (7*a*) hold for free and fixed† boundary spins, respectively.

In this letter the Ising strip with antiperiodic boundaries is considered. These boundary conditions induce a domain wall in the system for $T<T_c$ in the limit $L\to\infty$ and it is interesting to see how the correlations at $T=T_c$ with finite L are affected. Antiperiodic boundary conditions may be imposed by inserting an antiferromagnetic seam in a strip with periodic boundaries (Onsager 1944). Representing the insertion with disorder operators μ (Kadanoff and Ceva 1971) and placing the antiferromagnetic seam along the line $v=0$, one has (Cardy 1984c)

$$\langle\sigma(w_1)\sigma(w_2)\rangle_{\mathrm{A}}=\lim_{\substack{u_3\to\infty\\ u_4\to-\infty\\ v_3=v_4=0}}\frac{\langle\sigma(w_1)\sigma(w_2)\mu(w_3)\mu(w_4)\rangle_{\mathrm{P}}}{\langle\mu(w_3)\mu(w_4)\rangle_{\mathrm{P}}}. \tag{8}$$

Here the subscripts A and P refer to antiperiodic and periodic boundaries, respectively. Making use of the short-distance expansion of the $\sigma\mu$ products in equation (8) and of the scaling indices predicted by conformal invariance, Cardy (1984c) has shown that the correlation function on the left-hand side of (8) decays exponentially for $|u_1-u_2|\gg L$ with correlation length $\xi_{\mathrm{A}}=4L/3\pi$, in agreement with results of Burkhardt and Guim (1985) and Gehlen *et al* (1984, 1985).

In this letter the complete spatial dependence of $\langle\sigma(w_1)\sigma(w_2)\rangle_{\mathrm{A}}$ is exhibited. It follows directly from the result

$$\langle\sigma(z_1)\sigma(z_2)\mu(z_3)\mu(z_4)\rangle_\infty$$
$$=\pm\frac{A^2}{\sqrt{2}}\left|\left|\frac{z_{13}z_{24}}{z_{12}z_{14}z_{23}z_{34}}\right|^{1/2}+\left|\frac{z_{14}z_{23}}{z_{12}z_{13}z_{24}z_{34}}\right|^{1/2}-\left|\frac{z_{12}z_{34}}{z_{13}z_{14}z_{23}z_{24}}\right|^{1/2}\right|^{1/2} \tag{9}$$

with $z_{12}=z_1-z_2$, etc, of Belavin *et al* (1984) for the bulk critical-point correlations of two spin and two disorder operators in the Ising model. The sign of the correlation function (9) depends on the path followed by the antiferromagnetic seam between points z_3 and z_4, with a change in sign as the path is deformed through the spin variable at z_1 or z_2 (Kadanoff and Ceva 1971). The correlation function vanishes for finite $z_1,\ldots,z_4$ if

$$|z_{13}z_{24}|+|z_{14}z_{23}|-|z_{12}z_{34}|=0. \tag{10}$$

† The spins on both edges of the strip are fixed in the same state.

This condition is fulfilled if the quadrilateral formed by points $z_1, \ldots, z_4$ in the z plane is inscribed in a circle and has its diagonals between points 1 and 2 and points 3 and 4, as shown in figure 1.

The bulk two-point correlation function $\langle\mu(z_1)\mu(z_2)\rangle_\infty$ is given by the right-hand side of equation (3), as follows from the duality of the spin and disorder operators. The strip correlation functions on the right-hand side of (8) may be calculated from the corresponding bulk correlation functions just discussed with the help of equations (1) and (2). Inserting the results into equation (8), one obtains

$$\begin{aligned}\langle\sigma(w_1)\sigma(w_2)\rangle_{\mathrm{A}} \\ &= \pm\langle\sigma(w_1)\sigma(w_2)\rangle_{\mathrm{P}}\{\cosh(\pi(u_1-u_2)/L) \\ &\quad -[\cosh^2(\pi(u_1-u_2)/L)-\cos^2(\pi(v_1-v_2)/L)]^{1/2}\}^{1/2}\end{aligned} \tag{11}$$

for the spin–spin correlation function in an Ising strip with antiperiodic boundary conditions. The correlation function for periodic boundaries, which appears in equation (11), is given explicitly by equation (4) with $\eta=\frac{1}{4}$.

Equation (11) is the principal result of this letter. For an antiferromagnetic seam along the line $v=0$ and for v_1 and v_2 in the interval $0<v_1$, $v_2<L$, the $\pm$sign is to be interpreted so that $\langle\sigma(w_1)\sigma(w_2)\rangle_{\mathrm{A}}$ is positive for $|v_1-v_2|<L/2$ and negative for $L/2<|v_1-v_2|<L$. The correlation function, which is an odd function of the variable $L/2-|v_1-v_2|$, vanishes at $|v_1-v_2|=L/2$, where it is a non-singular function of v_1-v_2. The only singular points of the correlation function within the strip boundaries are at $w_1=w_2$.

Two limiting cases of particular interest will now be considered. For $|u_1-u_2|\gg L$, equation (11) takes the form

$$\langle\sigma(w_1)\sigma(w_2)\rangle_{\mathrm{A}}\approx A\left(\frac{2\pi}{L}\right)^{1/4}\exp\left(-\frac{3\pi}{4L}|u_1-u_2|\right)\cos(\pi(v_1-v_2)/L). \tag{12}$$

The characteristic length $\xi_{\mathrm{A}}=4L/3\pi$ of the exponential decay is consistent with the results of Burkhardt and Guim (1985), Cardy (1984c) and Gehlen *et al* (1984, 1985). The corresponding characteristic length for periodic boundaries is $\xi_{\mathrm{P}}=3\xi_{\mathrm{A}}=4L/\pi$, as follows from equation (4) with $\eta=\frac{1}{4}$. Thus the correlations of distant spins are substantially reduced by the antiferromagnetic seam.

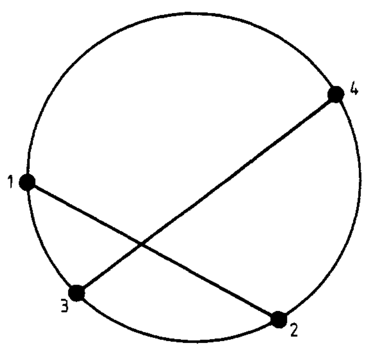

Figure 1. The four-point function (9) vanishes whenever the points z_1, z_2, z_3, z_4 are traversed in this order by a circle in the z plane.

In the special case $u_1-u_2=0$ corresponding to correlations perpendicular to the strip edge

$$\langle\sigma(w_1)\sigma(w_2)\rangle_{\mathrm{A}}=A\left[\frac{L}{\pi}\sin\left(\frac{\pi}{L}|v_1-v_2|\right)\right]^{-1/4}\left\{\sqrt{2}\sin\left[\frac{\pi}{2L}\left(\frac{L}{2}-|v_1-v_2|\right)\right]\right\}. \tag{13}$$

The antiferromagnetic seam contributes the factor in curly brackets in equation (13). This factor decreases smoothly from 1 to -1 as $|v_1-v_2|$ increases from 0 to L and changes sign at $|v_1-v_2|=L/2$.

It is interesting to compare the result (13) with the pair correlation function for a one-dimensional Ising system with antiperiodic boundaries, i.e. a ring of spins of circumference L with a single antiferromagnetic bond at $v=0$. For this system

$$\langle\sigma(v_1)\sigma(v_2)\rangle_{\mathrm{A}}^{d=1}=\sinh[(L/2-|v_1-v_2|)/\xi][\sinh(L/2\xi)]^{-1}. \tag{14}$$

Here ξ, the usual bulk correlation length, is given by

$$\xi=[\ln\coth(J/k_{\mathrm{B}}T)]^{-1}. \tag{15}$$

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